§ 4. Point Spectra in Shear Alfvén Modes

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In symmetric toroidal systems, the spectrum of shear Alfvén modes is continuous. It is easily understood by changing the equation of shear Alfvén modes:

$$\vec{B}\cdot\nabla\left[\frac{|\nabla\psi|^2}{B^2}\vec{B}\cdot\nabla\hat{\xi}\right]+\omega^2\rho_m\frac{|\nabla\psi|^2}{B^2}\hat{\xi}=0$$

into the Schrödinger form:

$$\begin{aligned} L\xi &= \frac{\partial^2\xi}{\partial\eta^2}+\left[\Omega^2\left(\frac{\langle B^2\rangle}{B^2}\right)^2-U(\eta)\right]\xi=0, \\ \Omega &= \Omega(\psi,\alpha),\;\; U(\eta) = \frac{1}{|\nabla\psi|}\frac{\partial^2|\nabla\psi|}{\partial\eta^2} \end{aligned}$$

The potential is exact periodic, and so, from the analogy to the Mathieu equation or Hill's equation, it can be concluded that only the continuous spectra exist. These are also understood from the investigation of the gap central frequency. Let $\Omega(\psi)$ be the frequency normalized by the local Alfvén frequency. Thus, the central gap frequency is determined by the condition:

$$\Omega(\psi)=|m-nq(\psi)|,\quad \Omega'=|m'-n'q(\psi)|$$

$$\left\{\begin{array}{l}\Omega(\psi)=\Omega'(\psi)\\ m'=m+m_{eq},\quad n'=n+n_{eq},\end{array}\right\}$$

leading to

$$\begin{aligned}\Omega(\psi) &= \frac{1}{2}|m_{eq}-n_{eq}q(\psi)|,\\ q_{min}\le q(\psi) &= \frac{2m+m_{eq}}{2n+n_{eq}}\le q_{max}.\end{aligned}$$

For the frequency range of TAE, $(m_{eq},n_{eq})=(\neq 0,0)$ and the difference of the neighboring central gap frequency is $\Delta\Omega_{gap}=1/2$, and for the frequency range of HAE, $(m_{eq},n_{eq})=(kM,kN)$ and $\Delta\Omega_{gap}=|M-Nq(\psi)|/2$.

This conclusion might still hold in the frequency range of TAE (Toroidicity-induced shear Alfén Eigenmodes) in a realistic three-dimensional configuration like LHD. In the frequency range of TAE, the structure of the spectrum gaps is almost determined by the axisymmetric components of the MHD equilibrium quantities. Also, in the frequency range of HAE(Helicity-induced shear Alfén Eigenmodes), the same situation might hold, because the spectrum of shear Alfén modes mainly determined helical symmetric components. Thus, the point spectra only appear when the local potential structure is introduced. However, when those typical frequency ranges become overlapping, the situation changes. In this case, the potential of the Schrödinger equation becomes a quasi-periodic, so that there appears a possibility for the point spectra to exist. This fact is confirmed by using the Leapunov exponent $\Lambda_\phi(\Omega)$ and the winding number $w(\Omega)$.

$$\begin{aligned}\text{phase} &: \tan\phi\equiv\frac{\xi}{\dot{\xi}}\\ \text{amplitude} &: A\equiv\sqrt{\xi^2+\dot{\xi}^2}\end{aligned}$$

Equation of the phase is

$$\frac{d\phi}{d\eta}=\frac{1+V(\eta)}{2}+\frac{1-V(\eta)}{2}\cos(2\phi)$$

Thus, the linearized version is

$$\frac{d\ln\delta\phi}{d\eta}=-[1-V(\eta)]\sin(2\phi).$$

Note $\delta\phi A^2=const$, and

$$\begin{aligned}V(\eta) &\equiv \Omega^2\left(\frac{\langle B^2\rangle}{B^2}\right)^2-U(\eta),\\ \Lambda_\phi(\Omega) &\equiv \lim_{\eta\to\infty}\frac{\ln\delta\phi(\eta)-\ln\delta\phi(\eta_0)}{\eta}\\ w(\Omega) &\equiv \lim_{\eta\to\infty}\frac{\phi(\eta)}{\eta}\end{aligned}$$

Note w is a continuous non-decreasing function of Ω with plateaus (like a devil's staircase). Classification of eigenvalues is known, from which the existence of point spectra is confirmed.